Analytical solutions of bound timelike geodesic orbits in Kerr spacetime

Ryuichi Fujita[1,2] and Wataru Hikida[2]

[1]Theoretical Physics, Raman Research Institute, Bangalore 560 080, India
[2]Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail: `draone@rri.res.in`

Abstract. We derive the analytical solutions of the bound timelike geodesic orbits in Kerr spacetime. The analytical solutions are expressed in terms of the elliptic integrals using Mino time λ as the independent variable. Mino time decouples the radial and polar motion of a particle and hence leads to forms more useful to estimate three fundamental frequencies, radial, polar and azimuthal motion, for the bound timelike geodesics in Kerr spacetime. This paper gives the first derivation of the analytical expressions of the fundamental frequencies. This paper also gives the first derivation of the analytical expressions of all coordinates for the bound timelike geodesics using Mino time. These analytical expressions should be useful not only to investigate physical properties of Kerr geodesics but more importantly to applications related to the estimation of gravitational waves from the extreme mass ratio inspirals.

1. Introduction

The Kerr black hole has been well studied since the discovery of the Kerr solution. It is an important topic not only in mathematical problems of general theory of relativity, but also for applications in astrophysics. Currently, there are many candidates for black holes in the universe and they have a wide range of mass scales ranging from stellar mass scales to galactic nuclei mass scales [1].

One of the ways to investigate the properties of a Kerr black hole spacetime is to study geodesic motion in this background. Detailed works on the geodesic motion in black hole spacetimes are summarized in Chandrasekhar [2]. In the weak field regime, at large distances from the black hole, the orbits of a particle are almost the same as that in Newtonian gravity. In the strong field regime, however, the orbits become more complicated and it is difficult to compare the orbits with that in Newtonian gravity. For the case of bound geodesics, this can be explained by mismatches between the fundamental frequencies of radial, Ω_r, polar, Ω_θ and azimuthal-motion, Ω_ϕ. For example, $\Omega_\phi - \Omega_\theta$ shows the precession of the orbital plane and $\Omega_\phi - \Omega_r$ shows the precession of the orbital ellipse. Differences between the fundamental frequencies become larger as the particle goes into the strong gravity region around black hole horizon or separatrix, which is the boundary between stable and unstable orbits. These relativistic effects have been studied for some cases and some examples of extreme phenomena are found as follows.

Wilkins [3] derived the analytical expressions for the ratio of the azimuthal frequency and the polar frequency, $\Omega_\phi/\Omega_\theta$, when a particle moves on both circular and non-equatorial orbits around the extreme Kerr black hole. He then showed that the ratio becomes larger as the particle approaches the horizon and found that the particle traces out a helix-like orbit on a sphere around the black hole. He also pointed out that there exist horizon-skimming orbits which have the same radius as the horizon. Horizon-skimming orbits are also studied by numerical calculations including the effects of the emission of gravitational waves from a particle for circular and non-equatorial orbits [4] and for generic orbits [5] around near-extremal Kerr black holes. Glampedakis and Kennefick [6] numerically investigated the ratio of the azimuthal frequency and the radial frequency, Ω_ϕ/Ω_r, when a particle moves both on eccentric and equatorial orbits around the Kerr black hole. They found that the ratio becomes larger as the particle approaches the separatrix and the particle traces out a quasi-circular orbit around the periapsis before going back to the apoapsis. These orbits are called zoom-whirl orbits.

The above results show that the fundamental frequencies play an important role in understanding bound geodesic orbits. However, the coupling of the r and θ motions in the geodesic equation has prevented one from deriving the fundamental frequencies, Ω_r, Ω_θ and Ω_ϕ, for general bound geodesic orbits until recently. Using the elegant Hamilton-Jacobi formalism, Schmidt [7] derived the fundamental frequencies without discussing the coupling of the r and θ motions. Although his results show that we can expand an arbitrary function of the particle's orbit in a Fourier series, we can not estimate the

Fourier components because of the coupling of the r and θ-motion. Mino [8] showed that we can separate r and θ-motion if we use new time parameter λ and derived the integral forms of the periods of both r and θ-motion with respect to λ, which is called Mino time. Combining Schmidt's method with Mino time, Drasco and Hughes [9] derived the fundamental frequencies and showed how the Fourier components of arbitrary functions of orbits with respect to Mino time can be computed because of the decoupling of both r and θ motions. They also showed how from these results using Mino time, the Fourier components with respect to coordinate time can also be derived. Thanks to these results, one can compute gravitational waves from binary systems in which a stellar mass compact star is moving on a general bound geodesic orbit around a supermassive black hole, the so-called extreme mass ratio inspirals(EMRIs) [10]. Gravitational waves from EMRIs are one of the main targets for space-based Laser Interferometer Space Antenna (LISA) [11].

In this paper, we derive analytical expressions for bound timelike geodesic orbits in Kerr spacetime using Mino time as the independent variable. Despite a lot of works on geodesic motion [2], the analytical expressions of null or timelike geodesics in Kerr spacetime are still important subjects. Fast and accurate computation of null geodesics in Kerr spacetime is required to study radiation which pass near black holes in accretion systems such as active galactic nuclei and X-ray binaries (see, for example, [12, 13] and references therein). Fast and accurate computation of timelike geodesics is also required to study gravitational waves from EMRIs and construct efficient templates for LISA data analysis. Rauch and Blandford gave tables which reduce the some integral forms of null geodesics to Legendre elliptic integrals [14] using the radial coordinate as the independent variable [12]. They did not give the complete tables which reduce all the integral forms to the elliptic integrals because it was easier and faster to compute both t and ϕ coordinates numerically when they studied the optical structure of the primary caustic around Kerr black hole. Using Carlson elliptic integrals [15] to calculate all coordinates of null geodesics, however, Dexter and Agol showed that they can compute null geodesics more efficiently than numerical integration method [13]. Although they did not show analytical expressions of all coordinates of null geodesics since there are so many cases to be considered, they opened their numerical code to compute null geodesics semi-analytically in Kerr spacetime. In this paper, we show that we can easily derive the analytical expressions of bound timelike geodesics in terms of Legendre elliptic integrals if we properly transform the r and θ variables. This is the first time that the analytical expressions of fundamental frequencies are derived. This is also the first time that the analytical expressions of all geodesic coordinates are derived using Mino time as the independent variable. These analytical expressions of bound timelike geodesic orbits with respect to Mino time are simpler than that in [13] for null geodesics and should be useful to investigate gravitational waves from EMRIs. The analytical solutions should also be helpful for investigations of bound geodesics in Kerr spacetime.

This paper is organized as follows. In section 2, we review Kerr geodesics using observer time. We then discuss Kerr geodesics in Mino time and derive the analytical

expressions of the fundamental frequencies of bound geodesics in section 3. In section 4, we derive the analytical expressions for bound geodesic orbits. We conclude with a brief summary in section 5. In the Appendices, we discuss technical details of the implementation required to obtain the results in this paper. Throughout this paper, we use units with $G = c = 1$.

2. Geodesic Orbits in Kerr Spacetime

The geodesic equations that describe a particle's orbits in Kerr spacetime are given by

$$\Sigma^2 \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = R(r),$$

$$\Sigma^2 \left(\frac{\mathrm{d}\cos\theta}{\mathrm{d}\tau}\right)^2 = \Theta(\cos\theta),$$

$$\Sigma\frac{\mathrm{d}t}{\mathrm{d}\tau} = T_{\mathrm{r}}(r) + T_\theta(\cos\theta) + a\mathcal{L}_z,$$

$$\Sigma\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \Phi_{\mathrm{r}}(r) + \Phi_\theta(\cos\theta) - a\mathcal{E}. \tag{1}$$

The functions $R(r)$, $\Theta(\cos\theta)$, $T_{\mathrm{r}}(r)$, $T_\theta(\cos\theta)$, $\Phi_{\mathrm{r}}(r)$ and $\Phi_\theta(\cos\theta)$ are defined by

$$R(r) = [P(r)]^2 - \Delta[r^2 + (a\mathcal{E} - \mathcal{L}_z)^2 + \mathcal{C}],$$

$$\Theta(\cos\theta) = \mathcal{C} - (\mathcal{C} + a^2(1 - \mathcal{E}^2) + \mathcal{L}_z^2)\cos^2\theta + a^2(1 - \mathcal{E}^2)\cos^4\theta,$$

$$T_r(r) = \frac{r^2 + a^2}{\Delta}P(r), \qquad T_\theta(\cos\theta) = -a^2\mathcal{E}(1 - \cos^2\theta),$$

$$\Phi_r(r) = \frac{a}{\Delta}P(r), \qquad \Phi_\theta(\cos\theta) = \frac{\mathcal{L}_z}{1 - \cos^2\theta},$$

with $P(r) = \mathcal{E}(r^2 + a^2) - a\mathcal{L}_z$, $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Here M and a are the mass and the angular momentum of the black hole, respectively. There are three constants of motion, $\mathcal{E}$, $\mathcal{L}_z$ and $\mathcal{C}$, which are the energy, the z-component of the angular momentum and the Carter constant per unit mass, respectively. Using reasonable initial conditions for the particle's orbit, we can derive the orbits using the proper time of the particle, τ, by numerical integration. Dividing $\mathrm{d}r/\mathrm{d}\tau$, $\mathrm{d}\cos\theta/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$ by $\mathrm{d}t/\mathrm{d}\tau$, we can also derive the orbits with coordinate time t by numerical integration. When the orbits are bound to black hole, however, we have to take care of the turning points in the radial and the polar motion where the signs of $\mathrm{d}r/\mathrm{d}\tau$ and $\mathrm{d}\cos\theta/\mathrm{d}\tau$ change. These turning points correspond to periapsis and apoapsis for the radial motion, and θ_{min} and $\pi - \theta_{\mathrm{min}}$ for the polar motion, where θ_{min} is the minimum value of θ. We need smaller stepsizes to resolve the derivatives around turning points. We can avoid this problem by introducing new variables for the radial and polar motion, $r = pM/(1 + e\cos\psi)$ and $\cos\theta = \cos\theta_{\mathrm{min}}\cos\chi$, where p is semilatus rectum and e is eccentricity [9]. Using these new variables, ψ and χ, we can estimate the orbits accurately.

There exists three fundamental frequencies, Ω_r, Ω_θ and Ω_ϕ, for bound Kerr geodesics. However, it is difficult to estimate the fundamental frequencies using (1) because of the coupling of the r and θ-motions. For instance, we immediately face

a difficulty when we estimate Ω_r using $dr/dt = (dr/d\tau)(dt/d\tau)^{-1}$ because r and θ asynchronously pass their turning points.

3. Fundamental Frequencies of bound geodesics

We now proceed to derive the analytical expressions for the fundamental frequencies of bound geodesic orbits using Mino time. In section 3.1, we briefly describe the Kerr geodesics in Mino time and then show how to derive the analytical expressions for the fundamental frequencies in section 3.2 and section 3.3. In section 3.4, we will check the analytical expressions by comparing them with earlier literature.

3.1. Geodesics in Mino time

Using Mino time, $\lambda = \int d\tau/\Sigma$, the geodesic equations become

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r),$$
$$\left(\frac{d\cos\theta}{d\lambda}\right)^2 = \Theta(\cos\theta),$$
$$\frac{dt}{d\lambda} = T_r(r) + T_\theta(\cos\theta) + a\mathcal{L}_z,$$
$$\frac{d\phi}{d\lambda} = \Phi_r(r) + \Phi_\theta(\cos\theta) - a\mathcal{E}. \tag{2}$$

It should be noted that, in (2), $dr/d\lambda$ depends only on r and $d\cos\theta/d\lambda$ depends only on $\cos\theta$. Thus the equations for radial and polar motion are decoupled. For the bound orbits, $r(\lambda)$ and $\cos\theta(\lambda)$ become periodic functions which are independent of each other. The fundamental periods for the radial and polar motion, Λ_r and Λ_θ, with respect to λ are given by

$$\Lambda_r = 2\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{R(r)}}, \qquad \Lambda_\theta = 4\int_0^{\cos\theta_{\min}} \frac{d\cos\theta}{\sqrt{\Theta(\cos\theta)}}, \tag{3}$$

where

$$r_{\min} = \frac{pM}{1+e}, \qquad r_{\max} = \frac{pM}{1-e}, \qquad \theta_{\text{inc}} + (\text{sgn}\,\mathcal{L}_z)\,\theta_{\min} = \frac{\pi}{2}. \tag{4}$$

Here $r_{\min}$ and $r_{\max}$ are the periapsis and apoapsis for the radial motion respectively, and θ_{inc} the inclination angle from the equatorial plane of black hole. Of course, $(\mathcal{E}, \mathcal{L}_z, \mathcal{C})$ are described by these orbital parameters $(p, e, \theta_{\text{inc}})$ and given in [7, 9]. The angular frequencies of the radial and the polar motion then become

$$\Upsilon_r = \frac{2\pi}{\Lambda_r}, \qquad \Upsilon_\theta = \frac{2\pi}{\Lambda_\theta}. \tag{5}$$

We also note that both $dt/d\lambda$ and $d\phi/d\lambda$ in (2) are the sum of a function of r and a function of $\cos\theta$. Then each equations are integrated as

$$t(\lambda) = \Gamma\lambda + t^{(r)}(\lambda) + t^{(\theta)}(\lambda), \qquad \phi(\lambda) = \Upsilon_\phi\lambda + \phi^{(r)}(\lambda) + \phi^{(\theta)}(\lambda), \tag{6}$$

where Γ and Υ_ϕ are the frequencies of coordinate time t and ϕ with respect to λ respectively, which are given by

$$
\begin{aligned}
\Gamma &= \Upsilon_{t^{(r)}} + \Upsilon_{t^{(\theta)}} + a\mathcal{L}_z, &\qquad \Upsilon_\phi &= \Upsilon_{\phi^{(r)}} + \Upsilon_{\phi^{(\theta)}} - a\mathcal{E}, \\
\Upsilon_{t^{(r)}} &= \langle T_r(r) \rangle_\lambda, &\qquad \Upsilon_{t^{(\theta)}} &= \langle T_\theta(\cos\theta) \rangle_\lambda, \\
\Upsilon_{\phi^{(r)}} &= \langle \Phi_r(r) \rangle_\lambda, &\qquad \Upsilon_{\phi^{(\theta)}} &= \langle \Phi_\theta(\cos\theta) \rangle_\lambda,
\end{aligned} \tag{7}
$$

where $\langle \cdots \rangle_\lambda \equiv \lim_{\Delta\lambda\to\infty} (2\Delta\lambda)^{-1} \int_{-\Delta\lambda}^{\Delta\lambda} d\lambda \cdots$ represents infinite time average with respect to λ, and $t^{(r)/(\theta)}$ and $\phi^{(r)/(\theta)}$ satisfy

$$
\begin{aligned}
\frac{dt^{(r)}}{d\lambda} &= T_r(r) - \Upsilon_{t^{(r)}}, &\qquad \frac{dt^{(\theta)}}{d\lambda} &= T_\theta(\cos\theta) - \Upsilon_{t^{(\theta)}}, \\
\frac{d\phi^{(r)}}{d\lambda} &= \Phi_r(r) - \Upsilon_{\phi^{(r)}}, &\qquad \frac{d\phi^{(\theta)}}{d\lambda} &= \Phi_\theta(\cos\theta) - \Upsilon_{\phi^{(\theta)}}.
\end{aligned} \tag{8}
$$

Equation (6) shows that both $t(\lambda)$ and $\phi(\lambda)$ consist of two distinct parts. The first term represents an accumulation over λ-time and the last two terms represent oscillations around it with periods $2\pi/\Upsilon_r$ and $2\pi/\Upsilon_\theta$. We note that the frequencies with respect to λ are related to the frequencies with distant observer time as [9]

$$
\Omega_r = \frac{\Upsilon_r}{\Gamma}, \qquad \Omega_\theta = \frac{\Upsilon_\theta}{\Gamma}, \qquad \Omega_\phi = \frac{\Upsilon_\phi}{\Gamma}. \tag{9}
$$

In the following subsections, section 3.2 and section 3.3, we discuss the analytical expressions for these frequencies. And we discuss the analytical expressions of the orbits, $r(\lambda)$, $\cos\theta(\lambda)$, $t(\lambda)$ and $\phi(\lambda)$, in section 4.

3.2. Frequencies of r and θ-motion

In this subsection, we derive the analytical expressions for the frequencies of r and θ-motion, Υ_r and Υ_θ, using (3). As explained in section 2, $R(r)$ and $\Theta(\cos\theta)$ become zero when r and $\cos\theta$ go through the turning points, $r_{\min}$, $r_{\max}$ and $\pm\cos\theta_{\min}$, respectively. Thus we usually transform the variables, r and $\cos\theta$, to avoid divergences in the numerical calculation. However we know that (3) can be expressed in terms of the elliptic integrals since both $R(r)$ and $\Theta(\cos\theta)$ are fourth order polynomials [16]. It is useful if we know the four zero points of both $R(r)$ and $\Theta(\cos\theta)$ to express (3) in terms of the elliptic integrals. We rewrite $R(r)$ and $\Theta(\cos\theta)$ as [9]

$$
\begin{aligned}
R(r) &= (1 - \mathcal{E}^2)(r_1 - r)(r - r_2)(r - r_3)(r - r_4), \\
\Theta(\cos\theta) &= \mathcal{L}_z^2 \epsilon_0 (z_- - \cos^2\theta)(z_+ - \cos^2\theta),
\end{aligned} \tag{10}
$$

where

$$
r_1 = \frac{pM}{1-e}, \qquad r_2 = \frac{pM}{1+e}, \qquad r_3 = \frac{(A+B) + \sqrt{(A+B)^2 - 4AB}}{2}, \qquad r_4 = \frac{AB}{r_3},
$$

$$
A + B = \frac{2M}{1 - \mathcal{E}^2} - (r_1 + r_2), \qquad AB = \frac{a^2\mathcal{C}}{(1 - \mathcal{E}^2) r_1 r_2}, \tag{11}
$$

and where $\epsilon_0 = a^2(1 - \mathcal{E}^2)/\mathcal{L}_z^2$, $z_- = \cos^2\theta_{\min}$ and $z_+ = \mathcal{C}/(\mathcal{L}_z^2 \epsilon_0 z_-)$. We note that two zero points, r_1 and r_2, of $R(r)$ are apoapsis and periapis respectively and two zero points, z_- and $-z_-$, of $\Theta(\cos\theta)$ are $\theta_{\min}$ and $\pi - \theta_{\min}$ respectively. These zero points

correspond to turning points, defined in (4), of radial and polar motion. But the other two zero points of both $R(r)$ and $\Theta(\cos\theta)$, r_3, r_4 and $\pm z_+$, do not correspond to turning points of radial and polar motion.

Using (10), we can express (3) in terms of the elliptic integrals as

$$
\int_{r_2}^{r} \frac{\mathrm{d}r'}{\sqrt{R(r')}} = \frac{2}{\sqrt{(1-\mathcal{E}^2)(r_1-r_3)(r_2-r_4)}} F(\arcsin y_r, k_r),
$$
$$
\int_{0}^{\cos\theta} \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} = \frac{1}{\mathcal{L}_z\sqrt{\epsilon_0 z_+}} F(\arcsin y_\theta, k_\theta), \tag{12}
$$

where

$$
y_r = \sqrt{\frac{r_1-r_3}{r_1-r_2}\frac{r-r_2}{r-r_3}}, \qquad k_r = \sqrt{\frac{r_1-r_2}{r_1-r_3}\frac{r_3-r_4}{r_2-r_4}},
$$
$$
y_\theta = \frac{\cos\theta}{\sqrt{z_-}}, \qquad k_\theta = \sqrt{\frac{z_-}{z_+}}, \tag{13}
$$

and $F(\varphi, k)$ is the incomplete elliptic integral of the first kind defined by

$$
F(\varphi, k) = \int_0^\varphi \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\varphi} \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-k^2y^2)}}. \tag{14}
$$

In the following, we describe both the elliptic integrals and the elliptic functions using the notation in [17]. The orbital frequencies of radial and polar motion with respect to λ are then given by

$$
\Upsilon_r = \frac{\pi\sqrt{(1-\mathcal{E}^2)(r_1-r_3)(r_2-r_4)}}{2K(k_r)}, \qquad \Upsilon_\theta = \frac{\pi\mathcal{L}_z\sqrt{\epsilon_0 z_+}}{2K(k_\theta)}. \tag{15}
$$

Here $K(k)$ is the complete elliptic integral of the first kind defined by $K(k) = F(\pi/2, k)$. We note that the analysis of θ-motion here is similar to that of Drasco and Hughes [9]. Though they used a different transformation of $\cos\theta$, our expression for Υ_θ in this subsection agrees with their final result.

3.3. Frequencies of t and ϕ-motion

In this subsection, we derive the analytical expressions for the frequencies of t and ϕ-motion, Γ and Υ_ϕ, using (7). Since the r and θ-motion decouple in Mino time, we can rewrite the infinite time average in (7) as an average over an orbital period, Λ_r or Λ_θ, as

$$
\Upsilon_{t(r)} = \frac{2}{\Lambda_r}\int_{r_2}^{r_1}\frac{T_r(r)}{\sqrt{R(r)}}\mathrm{d}r, \qquad \Upsilon_{t(\theta)} = \frac{4}{\Lambda_\theta}\int_0^{\sqrt{z_-}}\frac{T_\theta(\cos\theta)}{\sqrt{\Theta(\cos\theta)}}\mathrm{d}\cos\theta,
$$
$$
\Upsilon_{\phi(r)} = \frac{2}{\Lambda_r}\int_{r_2}^{r_1}\frac{\Phi_r(r)}{\sqrt{R(r)}}\mathrm{d}r, \qquad \Upsilon_{\phi(\theta)} = \frac{4}{\Lambda_\theta}\int_0^{\sqrt{z_-}}\frac{\Phi_\theta(\cos\theta)}{\sqrt{\Theta(\cos\theta)}}\mathrm{d}\cos\theta. \tag{16}
$$

It is straightforward to express $\Upsilon_{t(\theta)}$ and $\Upsilon_{\phi(\theta)}$ in terms of the elliptic integrals if we use y_θ in (13).

$$
\Upsilon_{t(\theta)} = -\frac{2a^2\mathcal{E}\Upsilon_\theta}{\pi\mathcal{L}_z\sqrt{\epsilon_0 z_+}}\left[(1-z_+)K(k_\theta) + z_+ E(\frac{\pi}{2}, k_\theta)\right],
$$

$$\Upsilon_{\phi^{(\theta)}} = \frac{2\Upsilon_\theta}{\pi\sqrt{\epsilon_0 z_+}}\Pi(\frac{\pi}{2}, -z_-, k_\theta),\tag{17}$$

where $E(\varphi, k)$ is the incomplete elliptic integral of the second kind and $\Pi(\varphi, c, k)$ is the incomplete elliptic integral of the third kind defined by

$$\begin{aligned}E(\varphi, k) &= \int_0^\varphi \sqrt{1 - k^2 \sin^2\theta}\mathrm{d}\theta,\\ \Pi(\varphi, c, k) &= \int_0^\varphi \frac{\mathrm{d}\theta}{(1 + c\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.\end{aligned}\tag{18}$$

Note that, $E(\pi/2, k)$ is the complete elliptic integral of the second kind and $\Pi(\pi/2, c, k)$ is the complete elliptic integral of the third kind. In the followings, we describe $E(\pi/2, k)$ as $E(k)$, and $\Pi(\pi/2, c, k)$ as $\Pi(c, k)$.

On the other hand, we have to rewrite $T_r(r)$ and $\Phi_r(r)$ in order to express $\Upsilon_{t^{(r)}}$ and $\Upsilon_{\phi^{(r)}}$ in terms of the elliptic integrals. Performing partial fraction decomposition, we decompose $T_r(r)$ and $\Phi_r(r)$ as follows.

$$\begin{aligned}T_r(r) &= \mathcal{E}r^2 + 2M\mathcal{E}r + \frac{2M}{r_+ - r_-}\left\{\frac{(4M^2\mathcal{E} - a\mathcal{L}_z)r_+ - 2Ma^2\mathcal{E}}{r - r_+} - (+ \leftrightarrow -)\right\}\\ &\quad + (a^2 + 4M^2)\mathcal{E} - a\mathcal{L}_z \quad (|a| \neq M),\\ &= \mathcal{E}r^2 + 2M\mathcal{E}r + \frac{2M(4M^2\mathcal{E} - a\mathcal{L}_z)}{r - M} + \frac{2M^2(2M^2\mathcal{E} - a\mathcal{L}_z)}{(r - M)^2}\\ &\quad + (a^2 + 4M^2)\mathcal{E} - a\mathcal{L}_z \quad (|a| = M),\\ \Phi_r(r) &= \frac{a}{r_+ - r_-}\left\{\frac{2M\mathcal{E}r_+ - a\mathcal{L}_z}{r - r_+} - (+ \leftrightarrow -)\right\} + a\mathcal{E} \quad (|a| \neq M),\\ &= \frac{2Ma\mathcal{E}}{r - M} + \frac{a(2M^2\mathcal{E} - a\mathcal{L}_z)}{(r - M)^2} + a\mathcal{E} \quad (|a| = M),\end{aligned}\tag{19}$$

with $r_\pm = M \pm \sqrt{M^2 - a^2}$. From (13), we find $r = r_3 + (r_2 - r_3)/(1 - h_r y_r^2)$, where $h_r = (r_1 - r_2)/(r_1 - r_3)$. Then it is straightforward to compute $\int_{r_2}^r r'\mathrm{d}r'/\sqrt{R(r')}$, which is composed in $\Upsilon_{t^{(r)}}$, as

$$\int_{r_2}^r \frac{r'}{\sqrt{R(r')}}\mathrm{d}r' = \frac{2\left[r_3 F(\arcsin y_r, k_r) + (r_2 - r_3)\Pi(\arcsin y_r, -h_r, k_r)\right]}{\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\tag{20}$$

We show the results of the other terms of $\int_{r_2}^r T_r(r')\mathrm{d}r'/\sqrt{R(r')}$ and $\int_{r_2}^r \Phi_r(r')\mathrm{d}r'/\sqrt{R(r')}$ in Appendix A. Using results quoted there, we derive Γ and Υ_ϕ as

$$\begin{aligned}\Gamma = 4M^2\mathcal{E} &+ \frac{2a^2\mathcal{E}z_+\Upsilon_\theta}{\pi\mathcal{L}_z\sqrt{\epsilon_0 z_+}}[K(k_\theta) - E(k_\theta)] + \frac{2\Upsilon_r}{\pi\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\\ &\times \left\{\frac{\mathcal{E}}{2}[(r_3(r_1 + r_2 + r_3) - r_1 r_2)K(k_r)\right.\\ &\quad + (r_2 - r_3)(r_1 + r_2 + r_3 + r_4)\Pi(-h_r, k_r)]\\ &\quad + (r_1 - r_3)(r_2 - r_4)E(k_r) + 2M\mathcal{E}[r_3 K(k_r) + (r_2 - r_3)\Pi(-h_r, k_r)]\\ &\quad + \frac{2M}{r_+ - r_-}\left[\frac{(4M^2\mathcal{E} - a\mathcal{L}_z)r_+ - 2Ma^2\mathcal{E}}{r_3 - r_+}\left(K(k_r) - \frac{r_2 - r_3}{r_2 - r_+}\Pi(-h_+, k_r)\right)\right.\\ &\quad \left.\left. - (+ \leftrightarrow -)]\right\}\right\},\\ \Upsilon_\phi = \frac{2\Upsilon_\theta}{\pi\sqrt{\epsilon_0 z_+}} &\Pi(-z_-, k_\theta) + \frac{2a\Upsilon_r}{\pi(r_+ - r_-)\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\end{aligned}$$

$$\times \left\{ \frac{2M\mathcal{E}r_+ - a\mathcal{L}_z}{r_3 - r_+} \left[K(k_r) - \frac{r_2 - r_3}{r_2 - r_+} \Pi(-h_+, k_r) \right] - (+ \leftrightarrow -) \right\}, \qquad (21)$$

where $h_\pm = (r_1 - r_2)(r_3 - r_\pm)/[(r_1 - r_3)(r_2 - r_\pm)]$. Combining (9), (15) and (21), we can derive the orbital frequencies with respect to observer time, Ω_r, Ω_θ and Ω_ϕ. It should be noted that (21) is not valid for the case $|a| = M$ since there exists divergent terms in $1/(r_+ - r_-)$. We show the expressions of Γ and Υ_ϕ for the case $|a| = M$ in Appendix B.

3.4. Consistency check of the fundamental frequencies

We can compare the expressions for the fundamental frequencies, Ω_r, Ω_θ and Ω_ϕ, in this section with that in earlier literature for some limiting cases. We can compare Ω_ϕ for the case $e = 0$ and $\theta_{\mathrm{inc}} = 0$ with [18], Ω_ϕ/Ω_r in the case $a = 0$, $e \neq 0$ and $\theta_{\mathrm{inc}} = 0$ with [19], and both Ω_ϕ and Ω_θ in the case $e = 0$ and $\theta_{\mathrm{inc}} \neq 0$ with [20]. It is a good check on our results that our more general analytical expressions for the fundamental frequencies are consistent with earlier work for the limiting cases.

We can check our results for more general cases. In table 1 and table 2, we compare our results with [21] in which the analytical expressions of the orbital frequencies are derived in terms of both post-Newtonian and small eccentricity expansions up through $O(v^5, e^2)$, where $v^2 = M/p$. In table 1 and table 2, we check our results for the cases $a \neq M$ and $a = M$ respectively. We find that relative errors are always less than 10^{-4} when we compare the results for the cases $p = 100M$ and $0.01 \leq e \leq 0.09$. Since the fundamental frequencies in [21] are derived up through $O(v^5, e^2)$, these relative errors, less than 10^{-4}, show the consistency of our results with [21]. In table 3, we compare our results with numerical integration method for the eccentric and inclined orbits such that $p = 6M$, $e = 0.7$, $\theta_{\mathrm{inc}} = 20°$ and $a = 0.9M$ or $a = M$. In numerical integration method, we use the trapezium rule to compute the fundamental frequencies. We can compute very accurately if we use the trapezium rule for the numerical integration of a periodic function. Then we find that the analytical expressions of the fundamental frequencies in this section agree with the results of numerical integration method. The relative errors are less than 10^{-15} in double precision calculation. These facts show that the analytical expressions in this section are correct in the cases of generic bound orbits.

4. Analytical solutions of bound geodesics

In this section, we derive the analytical expressions for bound geodesic orbits, $r(\lambda)$, $\cos\theta(\lambda)$, $t(\lambda)$ and $\phi(\lambda)$, in terms of the elliptic integrals. Since we have already derived the orbital frequencies in terms of the complete elliptic integrals in section 3, we can derive the orbits if we replace the complete elliptic integrals with the incomplete elliptic integrals. However, we have to take account of the initial values of both r and θ and the signs of both $\mathrm{d}r/\mathrm{d}\lambda$ and $\mathrm{d}\cos\theta/\mathrm{d}\lambda$ at given λ-time. In the following subsections, we derive the radial solutions, $r(\lambda)$, $t^{(r)}(\lambda)$ and $\phi^{(r)}(\lambda)$, in section 4.1, and the polar solutions, $\cos\theta(\lambda)$, $t^{(\theta)}(\lambda)$ and $\phi^{(\theta)}(\lambda)$, in section 4.2. Finally, we check the consistency of our analytical results by comparison with the results of earlier literature in section 4.3.

Table 1. Comparison of the orbital frequencies, Ω_r, Ω_θ and Ω_ϕ, derived using analytical expressions in this work and the analytical post-Newtonian expressions for orbits which are slightly eccentric but greatly inclined [21] in the case of $a = 0.9M$ and $p = 100M$. Our results are consistent with post-Newtonian results. Relative errors of the orbital frequencies are always less than 10^{-4}.

e	θ_{inc}	$\Omega_r^{\text{This work}}$	$\Omega_\theta^{\text{This work}}$	$\Omega_\phi^{\text{This work}}$	$\Omega_r^{\text{Post-Newton}}$	$\Omega_\theta^{\text{Post-Newton}}$	$\Omega_\phi^{\text{Post-Newton}}$
0.01	$20°$	9.71944×10^{-4}	9.97417×10^{-4}	9.99102×10^{-4}	9.71979×10^{-4}	9.97416×10^{-4}	9.99102×10^{-4}
0.01	$45°$	9.71338×10^{-4}	9.97974×10^{-4}	9.99688×10^{-4}	9.71367×10^{-4}	9.97975×10^{-4}	9.99689×10^{-4}
0.01	$70°$	9.70356×10^{-4}	9.98889×10^{-4}	1.00065×10^{-3}	9.70379×10^{-4}	9.98891×10^{-4}	1.00065×10^{-3}
0.05	$20°$	9.68513×10^{-4}	9.93895×10^{-4}	9.95575×10^{-4}	9.68558×10^{-4}	9.93905×10^{-4}	9.95586×10^{-4}
0.05	$45°$	9.67910×10^{-4}	9.94452×10^{-4}	9.96160×10^{-4}	9.67949×10^{-4}	9.94464×10^{-4}	9.96173×10^{-4}
0.05	$70°$	9.66934×10^{-4}	9.95366×10^{-4}	9.97118×10^{-4}	9.66968×10^{-4}	9.95380×10^{-4}	9.97132×10^{-4}
0.09	$20°$	9.60520×10^{-4}	9.85694×10^{-4}	9.87360×10^{-4}	9.60669×10^{-4}	9.85810×10^{-4}	9.87478×10^{-4}
0.09	$45°$	9.59926×10^{-4}	9.86249×10^{-4}	9.87943×10^{-4}	9.60069×10^{-4}	9.86368×10^{-4}	9.88063×10^{-4}
0.09	$70°$	9.58963×10^{-4}	9.87162×10^{-4}	9.88899×10^{-4}	9.59101×10^{-4}	9.87282×10^{-4}	9.89020×10^{-4}

Table 2. Comparison of the orbital frequencies, Ω_r, Ω_θ and Ω_ϕ, derived using analytical expressions in this work and the analytical post-Newtonian expressions for orbits which are slightly eccentric but greatly inclined [21] in the case of $a = M$ and $p = 100M$. Our results are consistent with post-Newtonian results. Relative errors of the orbital frequencies are always less than 10^{-4}.

e	θ_{inc}	$\Omega_r^{\text{This work}}$	$\Omega_\theta^{\text{This work}}$	$\Omega_\phi^{\text{This work}}$	$\Omega_r^{\text{Post-Newton}}$	$\Omega_\theta^{\text{Post-Newton}}$	$\Omega_\phi^{\text{Post-Newton}}$
0.01	$20°$	9.72213×10^{-4}	9.97159×10^{-4}	9.99017×10^{-4}	9.72250×10^{-4}	9.97157×10^{-4}	9.99017×10^{-4}
0.01	$45°$	9.71546×10^{-4}	9.97769×10^{-4}	9.99663×10^{-4}	9.71575×10^{-4}	9.97770×10^{-4}	9.99664×10^{-4}
0.01	$70°$	9.70462×10^{-4}	9.98777×10^{-4}	1.00073×10^{-3}	9.70484×10^{-4}	9.98780×10^{-4}	1.00073×10^{-3}
0.05	$20°$	9.68780×10^{-4}	9.93637×10^{-4}	9.95490×10^{-4}	9.68827×10^{-4}	9.93647×10^{-4}	9.95501×10^{-4}
0.05	$45°$	9.68117×10^{-4}	9.94247×10^{-4}	9.96134×10^{-4}	9.68157×10^{-4}	9.94259×10^{-4}	9.96147×10^{-4}
0.05	$70°$	9.67039×10^{-4}	9.95254×10^{-4}	9.97196×10^{-4}	9.67072×10^{-4}	9.95268×10^{-4}	9.97210×10^{-4}
0.09	$20°$	9.60784×10^{-4}	9.85436×10^{-4}	9.87273×10^{-4}	9.60935×10^{-4}	9.85553×10^{-4}	9.87391×10^{-4}
0.09	$45°$	9.60130×10^{-4}	9.86045×10^{-4}	9.87916×10^{-4}	9.60274×10^{-4}	9.86164×10^{-4}	9.88036×10^{-4}
0.09	$70°$	9.59067×10^{-4}	9.87050×10^{-4}	9.88975×10^{-4}	9.59204×10^{-4}	9.87171×10^{-4}	9.89097×10^{-4}

Table 3. Comparison of the orbital frequencies, Ω_r, Ω_θ and Ω_ϕ, derived using analytical expressions in this work and numerical integration method in the case of $p = 6M$, $e = 0.7$, $\theta_{\text{inc}} = 20°$ and $a = 0.9M$ or $a = M$. Our results are consistent with numerical integration method. Relative errors of the orbital frequencies agree with the accuracy of double precision calculation.

a/M	$\Omega_{r,\theta,\phi}$	This work	Numerical integration	Absolute value of relative error
0.9	Ω_r	$1.8928532285101992 \times 10^{-2}$	$1.8928532285101982 \times 10^{-2}$	5.6×10^{-16}
0.9	Ω_θ	$2.7299110395017517 \times 10^{-2}$	$2.7299110395017506 \times 10^{-2}$	4.1×10^{-16}
0.9	Ω_ϕ	$3.0550463796964692 \times 10^{-2}$	$3.0550463796964682 \times 10^{-2}$	3.6×10^{-16}
1	Ω_r	$1.9343466898960462 \times 10^{-2}$	$1.9343466898960444 \times 10^{-2}$	7.8×10^{-16}
1	Ω_θ	$2.6337035996626332 \times 10^{-2}$	$2.6337035996626321 \times 10^{-2}$	4.3×10^{-16}
1	Ω_ϕ	$2.9662029663040452 \times 10^{-2}$	$2.9662029663040452 \times 10^{-2}$	4.0×10^{-17}

4.1. Radial solution :$r(\lambda)$, $t^{(r)}(\lambda)$ and $\phi^{(r)}(\lambda)$

Solving (12) and (8), we obtain $\lambda(r)$, $t^{(r)}(\lambda)$ and $\phi^{(r)}(\lambda)$ as

$$
\begin{aligned}
\lambda(r) &= \int^r \frac{\mathrm{d}r'}{\sqrt{R(r')}}, \\
t^{(r)}(\lambda) &= \int^{r(\lambda)} \frac{T_r(r') - \Upsilon_{t^{(r)}}}{\sqrt{R(r')}} \mathrm{d}r', \\
\phi^{(r)}(\lambda) &= \int^{r(\lambda)} \frac{\Phi_r(r') - \Upsilon_{\phi^{(r)}}}{\sqrt{R(r')}} \mathrm{d}r'.
\end{aligned}
\tag{22}
$$

We derive $r(\lambda)$ inverting $\lambda(r)$. Since the period of r-motion with respect to λ is $\Lambda_r = 2\pi/\Upsilon_r$, we map λ to $\lambda^{(r)}$ as $\lambda^{(r)} = \lambda - 2\pi[\Upsilon_r\lambda/2\pi]/\Upsilon_r$, where $[\cdots]$ is the floor function, in the following subsections. In order to investigate the integrations in (22) properly, we have to take account of $r(\lambda = 0)$ and the sign of $\mathrm{d}r(\lambda)/\mathrm{d}\lambda$. There exist two cases depending on whether the initial value is $\mathrm{d}r(0)/\mathrm{d}\lambda \geq 0$ or $\mathrm{d}r(0)/\mathrm{d}\lambda \leq 0$. In the following subsections, we consider the two cases separately. We note that the expressions of both $t^{(r)}(\lambda)$ and $\phi^{(r)}(\lambda)$ in the following subsections are valid when $|a| \neq M$. We show $t^{(r)}(\lambda)$ and $\phi^{(r)}(\lambda)$ when $|a| = M$ in Appendix B.

4.1.1. $\mathrm{d}r(0)/\mathrm{d}\lambda \geq 0$ case In this subsection, we consider the case that the initial value of $r(\lambda)$ satisfy $\mathrm{d}r(0)/\mathrm{d}\lambda \geq 0$. We set $r(\lambda = 0) = r_0^{(1)}$ in this subsection. Then $\lambda(r)$ in (22) can be expressed as

$$
\begin{aligned}
\lambda^{(r)}(r) &= \int_{r_0^{(1)}}^r \frac{\mathrm{d}r'}{\sqrt{R(r')}}, \\
&= \left[\int_{r_2}^r - \int_{r_2}^{r_0^{(1)}}\right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} & r : r_0^{(1)} \to r_1, \\
&= \left[-\int_{r_2}^r + 2\int_{r_2}^{r_1} - \int_{r_2}^{r_0^{(1)}}\right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} & r : r_1 \to r_2, \\
&= \left[\int_{r_2}^r + 2\int_{r_2}^{r_1} - \int_{r_2}^{r_0^{(1)}}\right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} & r : r_2 \to r_0^{(1)}.
\end{aligned}
\tag{23}
$$

Thus we find the solution as

$$
\lambda^{(r)}(r) = \begin{cases}
\lambda_0^{(r)}(r) - \Lambda_r^{(1)} & r : r_0^{(1)} \to r_1, \\
-\lambda_0^{(r)}(r) + \Lambda_r - \Lambda_r^{(1)} & r : r_1 \to r_2, \\
\lambda_0^{(r)}(r) + \Lambda_r - \Lambda_r^{(1)} & r : r_2 \to r_0^{(1)},
\end{cases}
\tag{24}
$$

where

$$
\lambda_0^{(r)}(r) = \frac{1}{\sqrt{1 - \mathcal{E}^2}} \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}} F\left(\arcsin y_r, k_r\right),
\tag{25}
$$

and $\Lambda_r^{(1)} = \lambda_0^{(r)}(r_0^{(1)})$.

Inverting (24), we derive $r(\lambda)$ as

$$r(\lambda) = \frac{r_3(r_1 - r_2)\mathrm{sn}^2(u_r(\lambda), k_r) - r_2(r_1 - r_3)}{(r_1 - r_2)\mathrm{sn}^2(u_r(\lambda), k_r) - (r_1 - r_3)}, \tag{26}$$

where $\mathrm{sn}(u, k)$ is Jacobi's elliptic function which is defined as the inverse function of the incomplete elliptic integrals, $u = F(\varphi, k)$, and

$$u_r(\lambda) = \begin{cases} 2K(k_r)(\lambda^{(r)} + \Lambda_r^{(1)})/\Lambda_r & (0 \leq \lambda^{(r)} \leq \Lambda_r/2 - \Lambda_r^{(1)}), \\ 2K(k_r)(-\lambda^{(r)} + \Lambda_r - \Lambda_r^{(1)})/\Lambda_r & (\Lambda_r/2 - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r - \Lambda_r^{(1)}), \\ 2K(k_r)(\lambda^{(r)} - \Lambda_r + \Lambda_r^{(1)})/\Lambda_r & (\Lambda_r - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r). \end{cases} \tag{27}$$

Combining the results of (23) with the results of Appendix A, we can derive $t^{(r)}$ and $\phi^{(r)}$ in (22) as

$$
\begin{aligned}
t^{(r)} = {} & \frac{2}{\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}} \\
& \times \Bigg\{ \frac{\mathcal{E}}{2} \Big[(r_2 - r_3)(r_1 + r_2 + r_3 + r_4)\tilde{\Pi}_r(\psi_r, -h_r, k_r) \\
& + (r_1 - r_3)(r_2 - r_4)\tilde{E}_r(\psi_r, h_r, k_r) \Big] \\
& + 2M\mathcal{E}(r_2 - r_3)\tilde{\Pi}_r(\psi_r, -h_r, k_r) \\
& - \frac{2M}{r_+ - r_-} \Big[\frac{(4M^2\mathcal{E} - a\mathcal{L}_z)r_+ - 2Ma^2\mathcal{E}}{r_3 - r_+} \frac{r_2 - r_3}{r_2 - r_+} \tilde{\Pi}_r(\psi_r, -h_+, k_r) \\
& - (+ \leftrightarrow -)] \Bigg\}, \\
\phi^{(r)} = {} & -\frac{2a}{(r_+ - r_-)\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}} \Bigg[\frac{(2M\mathcal{E}r_+ - a\mathcal{L}_z)(r_2 - r_3)}{(r_3 - r_+)(r_2 - r_+)} \tilde{\Pi}_r(\psi_r, -h_+, k_r) \\
& - (+ \leftrightarrow -) \Bigg],
\end{aligned}
\tag{28}
$$

where $\psi_r = \arcsin[\mathrm{sn}(u_r, k_r)]$, $\tilde{E}_r(\psi_r, c, k_r) = E_r(\psi_r, c, k_r) - \frac{\Upsilon_r\lambda^{(r)}}{\pi}E(k_r)$, $\tilde{\Pi}_r(\psi_r, c, k_r) = \Pi_r(\psi_r, c, k_r) - \frac{\Upsilon_r\lambda^{(r)}}{\pi}\Pi(c, k_r)$ and

$$
\begin{aligned}
E_r^{(0)}(\psi_r, c, k_r) = {} & E(\psi_r, k_r) + \frac{\sin\psi_r\sqrt{(1 - \sin^2\psi_r)(1 - k_r^2\sin^2\psi_r)}}{\sin^2\psi_r - c^{-1}}, \\
E_r(\psi_r, c, k_r) = {} & E_r^{(0)}(\psi_r, c, k_r) - E_r^{(0)}(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad 0 \leq \lambda^{(r)} \leq \Lambda_r/2 - \Lambda_r^{(1)}, \\
= {} & - E_r^{(0)}(\psi_r, c, k_r) + 2E(k_r) - E_r^{(0)}(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad \Lambda_r/2 - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r - \Lambda_r^{(1)}, \\
= {} & E_r^{(0)}(\psi_r, c, k_r) + 2E(k_r) - E_r^{(0)}(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad \Lambda_r - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r, \\
\Pi_r(\psi_r, c, k_r) = {} & \Pi(\psi_r, c, k_r) - \Pi(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad 0 \leq \lambda^{(r)} \leq \Lambda_r/2 - \Lambda_r^{(1)}, \\
= {} & - \Pi(\psi_r, c, k_r) + 2\Pi(c, k_r) - \Pi(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad \Lambda_r/2 - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r - \Lambda_r^{(1)}, \\
= {} & \Pi(\psi_r, c, k_r) + 2\Pi(c, k_r) - \Pi(\psi_r(0), c, k_r) \\
& \qquad \text{for} \quad \Lambda_r - \Lambda_r^{(1)} \leq \lambda^{(r)} \leq \Lambda_r.
\end{aligned}
\tag{29}
$$

4.1.2. $\mathrm{d}r(0)/\mathrm{d}\lambda \leq 0$ case In this subsection, we consider the case that the initial value of $r(\lambda)$ satisfy $\mathrm{d}r(0)/\mathrm{d}\lambda \leq 0$. We set $r(\lambda = 0) = r_0^{(2)}$ in this subsection. Then $\lambda(r)$ in (22) can be expressed as

$$
\begin{aligned}
\lambda^{(r)}(r) &= \int_{r_0^{(2)}}^{r} \frac{\mathrm{d}r'}{\sqrt{R(r')}}, \\
&= \left[-\int_{r_2}^{r} + \int_{r_2}^{r_0^{(2)}} \right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} && r : r_0^{(2)} \to r_2, \\
&= \left[\int_{r_2}^{r} + \int_{r_2}^{r_0^{(2)}} \right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} && r : r_2 \to r_1, \\
&= \left[-\int_{r_2}^{r} + 2\int_{r_2}^{r_1} + \int_{r_2}^{r_0^{(2)}} \right] \frac{\mathrm{d}r'}{\sqrt{R(r')}} && r : r_1 \to r_0^{(2)}.
\end{aligned}
\tag{30}
$$

Thus we find the solution as

$$
\lambda^{(r)}(r) = \begin{cases}
-\lambda_0^{(r)}(r) + \Lambda_r^{(2)} & r : r_0^{(2)} \to r_2, \\
\lambda_0^{(r)}(r) + \Lambda_r^{(2)} & r : r_2 \to r_1, \\
-\lambda_0^{(r)}(r) + \Lambda_r + \Lambda_r^{(2)} & r : r_1 \to r_0^{(2)},
\end{cases}
\tag{31}
$$

where $\Lambda_r^{(2)} = \lambda_0^{(r)}(r_0^{(2)})$.

Then we obtain $r(\lambda)$ in the same form in (26) inverting (31), but $u_r(\lambda)$ in (26) is modified as

$$
u_r(\lambda) = \begin{cases}
2K(k_r)(-\lambda^{(r)} + \Lambda_r^{(2)})/\Lambda_r & (0 \leq \lambda^{(r)} \leq \Lambda_r^{(2)}), \\
2K(k_r)(\lambda^{(r)} - \Lambda_r^{(2)})/\Lambda_r & (\Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r/2 + \Lambda_r^{(2)}), \\
2K(k_r)(-\lambda^{(r)} + \Lambda_r + \Lambda_r^{(2)})/\Lambda_r & (\Lambda_r/2 + \Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r).
\end{cases}
\tag{32}
$$

Combining the results of (30) with the results of Appendix A, we can derive $t^{(r)}$ and $\phi^{(r)}$ in the same form in (28), but $E_r(\psi_r, c, k_r)$ and $\Pi_r(\psi_r, c, k_r)$ are modified as

$$
\begin{aligned}
E_r(\psi_r, c, k_r) &= -E_r^{(0)}(\psi_r, c, k_r) + E_r^{(0)}(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad 0 \leq \lambda^{(r)} \leq \Lambda_r^{(2)}, \\
&= E_r^{(0)}(\psi_r, c, k_r) + E_r^{(0)}(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad \Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r/2 + \Lambda_r^{(2)}, \\
&= -E_r^{(0)}(\psi_r, c, k_r) + 2E(k_r) + E_r^{(0)}(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad \Lambda_r/2 + \Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r, \\
\Pi_r(\psi_r, c, k_r) &= -\Pi(\psi_r, c, k_r) + \Pi(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad 0 \leq \lambda^{(r)} \leq \Lambda_r^{(2)}, \\
&= \Pi(\psi_r, c, k_r) + \Pi(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad \Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r/2 + \Lambda_r^{(2)}, \\
&= -\Pi(\psi_r, c, k_r) + 2\Pi(c, k_r) + \Pi(\psi_r(0), c, k_r) \\
&\qquad\qquad \text{for} \qquad \Lambda_r/2 + \Lambda_r^{(2)} \leq \lambda^{(r)} \leq \Lambda_r.
\end{aligned}
\tag{33}
$$

4.2. Polar Solution : $\cos\theta(\lambda)$, $t^{(\theta)}(\lambda)$ *and* $\phi^{(\theta)}(\lambda)$

Solving (12) and (8), we obtain $\lambda(\cos\theta)$, $t^{(\theta)}(\lambda)$ and $\phi^{(\theta)}(\lambda)$ as

$$
\begin{aligned}
\lambda(\theta) \quad &= \int^{\cos\theta} \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}}, \\
t^{(\theta)}(\lambda) &= \int^{\cos\theta} \frac{T_\theta(\cos\theta') - \Upsilon_{t^{(\theta)}}}{\sqrt{\Theta(\cos\theta')}} \mathrm{d}\cos\theta', \\
\phi^{(\theta)}(\lambda) &= \int^{\cos\theta} \frac{\Phi_\theta(\cos\theta') - \Upsilon_{\phi^{(\theta)}}}{\sqrt{\Theta(\cos\theta')}} \mathrm{d}\cos\theta'.
\end{aligned}
\tag{34}
$$

We derive $\cos\theta(\lambda)$ inverting $\lambda(\theta)$. Since the period of θ-motion with respect to λ is $\Lambda_\theta = 2\pi/\Upsilon_\theta$, we map λ to $\lambda^{(\theta)}$ as $\lambda^{(\theta)} = \lambda - 2\pi[\Upsilon_\theta\lambda/2\pi]/\Upsilon_\theta$ in the following subsections. In order to investigate the integrations in (34) properly, we have to take account of $\cos\theta(\lambda=0)$ and the sign of $\mathrm{d}\cos\theta(\lambda)/\mathrm{d}\lambda$. There exist two cases depending on whether the initial value is $\mathrm{d}\cos\theta(0)/\mathrm{d}\lambda \geq 0$ or $\mathrm{d}\cos\theta(0)/\mathrm{d}\lambda \leq 0$. In the following subsections, we consider the two cases separately.

4.2.1. $\mathrm{d}\cos\theta(0)/d\lambda \geq 0$ case In this subsection, we consider the case that the initial value of $\cos\theta(\lambda)$ satisfies $\mathrm{d}\cos\theta(0)/\mathrm{d}\lambda \geq 0$. We set $\theta(\lambda=0) = \theta_0^{(1)}$ in this subsection. Then $\lambda(\theta)$ in (34) can be expressed as

$$
\begin{aligned}
\lambda^{(\theta)}(\theta) \qquad &= \int_{\cos\theta_0^{(1)}}^{\cos\theta} \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}}, \\
&= \left[\int_0^{\cos\theta} - \int_0^{\cos\theta_0^{(1)}}\right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} \qquad &\theta : \theta_0^{(1)} \to \theta_{\min}, \\
&= \left[-\int_0^{\cos\theta} + 2\int_0^{\cos\theta_{\min}} - \int_0^{\cos\theta_0^{(1)}}\right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} \qquad &\theta : \theta_{\min} \to \pi - \theta_{\min}, \\
&= \left[\int_0^{\cos\theta} + 4\int_0^{\cos\theta_{\min}} - \int_0^{\cos\theta_0^{(1)}}\right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} \qquad &\theta : \pi - \theta_{\min} \to \theta_0^{(1)} (35)
\end{aligned}
$$

Thus we find the solution as

$$
\lambda^{(\theta)}(\theta) = \begin{cases}
\lambda_0^{(\theta)}(\theta) - \Lambda_\theta^{(1)} & \theta : \theta_0^{(1)} \to \theta_{\min}, \\
-\lambda_0^{(\theta)}(\theta) + \Lambda_\theta/2 - \Lambda_\theta^{(1)} & \theta : \theta_{\min} \to \pi - \theta_{\min}, \\
\lambda_0^{(\theta)}(\theta) + \Lambda_\theta - \Lambda_\theta^{(1)} & \theta : \pi - \theta_{\min} \to \theta_0^{(1)},
\end{cases}
\tag{36}
$$

where

$$
\lambda_0^{(\theta)}(\theta) = \frac{1}{\mathcal{L}_z\sqrt{\epsilon_0 z_+}} F\left(\arcsin y_\theta, k_\theta\right),
\tag{37}
$$

and $\Lambda_\theta^{(1)} = \lambda_0^{(\theta)}(\theta_0^{(1)})$.

Inverting (36), we derive $\cos\theta(\lambda)$ as

$$
\cos\theta(\lambda) = \sqrt{z_-}\mathrm{sn}(u_\theta(\lambda), k_\theta),
\tag{38}
$$

where

$$
u_\theta(\lambda) = \begin{cases}
4K(k_\theta)(\lambda^{(\theta)} + \Lambda_\theta^{(1)})/\Lambda_\theta & (0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 - \Lambda_\theta^{(1)}), \\
4K(k_\theta)(-\lambda^{(\theta)} + \Lambda_\theta/2 - \Lambda_\theta^{(1)})/\Lambda_\theta & (\Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 - \Lambda_\theta^{(1)}), \\
4K(k_\theta)(\lambda^{(\theta)} - \Lambda_\theta + \Lambda_\theta^{(1)})/\Lambda_\theta & (3\Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le \Lambda_\theta).
\end{cases}
$$

Using the results of (35), we derive $t^{(\theta)}$ and $\phi^{(\theta)}$ as

$$
t^{(\theta)} = \frac{a^2 \mathcal{E} z_+}{\mathcal{L}_z \sqrt{\epsilon_0 z_+}} \left[\frac{2\Upsilon_\theta \lambda^{(\theta)}}{\pi} E(k_\theta) - E_\theta(\psi_\theta, k_\theta) \right],
$$

$$
\phi^{(\theta)} = \frac{1}{\sqrt{\epsilon_0 z_+}} \left[\Pi_\theta(\psi_\theta, -z_-, k_\theta) - \frac{2\Upsilon_\theta \lambda^{(\theta)}}{\pi} \Pi(-z_-, k_\theta) \right], \tag{39}
$$

where $\psi_\theta = \arcsin[\operatorname{sn}(u_\theta, k_\theta)]$ and

$$
\begin{aligned}
E_\theta(\psi_\theta, k_\theta) \quad &= E(\psi_\theta, k_\theta) - E(\psi_\theta(0), k_\theta) \\
&\qquad\qquad \text{for} \quad 0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 - \Lambda_\theta^{(1)}, \\
&= -E(\psi_\theta, k_\theta) + 2E(k_\theta) - E(\psi_\theta(0), k_\theta) \\
&\qquad\qquad \text{for} \quad \Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 - \Lambda_\theta^{(1)}, \\
&= E(\psi_\theta, k_\theta) + 4E(k_\theta) - E(\psi_\theta(0), k_\theta) \\
&\qquad\qquad \text{for} \quad 3\Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le \Lambda_\theta, \tag{40}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_\theta(\psi_\theta, c, k_\theta) \quad &= \Pi(\psi_\theta, c, k_\theta) - \Pi(\psi_\theta(0), c, k_\theta) \\
&\qquad\qquad \text{for} \quad 0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 - \Lambda_\theta^{(1)}, \\
&= -\Pi(\psi_\theta, c, k_\theta) + 2\Pi(c, k_\theta) - \Pi(\psi_\theta(0), c, k_\theta) \\
&\qquad\qquad \text{for} \quad \Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 - \Lambda_\theta^{(1)}, \\
&= \Pi(\psi_\theta, c, k_\theta) + 4\Pi(c, k_\theta) - \Pi(\psi_\theta(0), c, k_\theta) \\
&\qquad\qquad \text{for} \quad 3\Lambda_\theta/4 - \Lambda_\theta^{(1)} \le \lambda^{(\theta)} \le \Lambda_\theta. \tag{41}
\end{aligned}
$$

4.2.2. $\mathrm{d}\cos\theta(0)/\mathrm{d}\lambda \le 0$ case In this subsection, we consider the case that the initial value of $\cos\theta(\lambda)$ satisfies $\mathrm{d}\cos\theta(0)/\mathrm{d}\lambda \le 0$. We set $\theta(\lambda = 0) = \theta_0^{(2)}$ in this subsection. Then $\lambda(\theta)$ in (34) can be expressed as

$$
\begin{aligned}
\lambda^{(\theta)}(\theta) \quad &= \int_{\cos\theta_0^{(2)}}^{\cos\theta} \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}}, \\
&= \left[-\int_0^{\cos\theta} + \int_0^{\cos\theta_0^{(2)}} \right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} & \theta : \theta_0^{(2)} \to \pi - \theta_{\min}, \\
&= \left[\int_0^{\cos\theta} + 2\int_0^{\cos\theta_{\min}} + \int_0^{\cos\theta_0^{(2)}} \right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} & \theta : \pi - \theta_{\min} \to \theta_{\min}, \\
&= \left[-\int_0^{\cos\theta} + 4\int_0^{\cos\theta_{\min}} + \int_0^{\cos\theta_0^{(2)}} \right] \frac{\mathrm{d}\cos\theta'}{\sqrt{\Theta(\cos\theta')}} & \theta : \theta_{\min} \to \theta_0^{(2)}. \tag{42}
\end{aligned}
$$

Thus we find the solution as

$$
\lambda^{(\theta)}(\theta) = \begin{cases}
-\lambda_0^{(\theta)}(\theta) + \Lambda_\theta^{(2)} & \theta : \theta_0^{(2)} \to \pi - \theta_{\min}, \\
\lambda_0^{(\theta)}(\theta) + \Lambda_\theta/2 + \Lambda_\theta^{(2)} & \theta : \pi - \theta_{\min} \to \theta_{\min}, \\
-\lambda_0^{(\theta)}(\theta) + \Lambda_\theta + \Lambda_\theta^{(2)} & \theta : \theta_{\min} \to \theta_0^{(2)},
\end{cases} \tag{43}
$$

where $\Lambda_\theta^{(2)} = \lambda_0^{(\theta)}(\theta_0^{(2)})$.

Then we obtain $\cos\theta(\lambda)$ in the same form in (38) inverting (43), but $u_\theta(\lambda)$ in (38) is modified as

$$u_\theta(\lambda) = \begin{cases} 4K(k_\theta)(-\lambda^{(\theta)} + \Lambda_\theta^{(2)})/\Lambda_\theta & (0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 + \Lambda_\theta^{(2)}), \\ 4K(k_\theta)(\lambda^{(\theta)} - \Lambda_\theta/2 - \Lambda_\theta^{(2)})/\Lambda_\theta & (\Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 + \Lambda_\theta^{(2)}), \\ 4K(k_\theta)(-\lambda^{(\theta)} + \Lambda_\theta + \Lambda_\theta^{(2)})/\Lambda_\theta & (3\Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le \Lambda_\theta). \end{cases}$$

Using the results of (42), we derive $t^{(\theta)}$ and $\phi^{(\theta)}$ as in the same form in (39), but $E_\theta(\psi_\theta, k_\theta)$ and $\Pi_\theta(\psi_\theta, c, k_\theta)$ are modified as

$$\begin{aligned} E_\theta(\psi_\theta, k_\theta) \quad &= -E(\psi_\theta, k_\theta) + E(\psi_\theta(0), k_\theta) \\ &\qquad \text{for} \qquad 0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 + \Lambda_\theta^{(2)}, \\ &= E(\psi_\theta, k_\theta) + 2E(k_\theta) + E(\psi_\theta(0), k_\theta) \\ &\qquad \text{for} \qquad \Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 + \Lambda_\theta^{(2)}, \\ &= -E(\psi_\theta, k_\theta) + 4E(k_\theta) + E(\psi_\theta(0), k_\theta) \\ &\qquad \text{for} \qquad 3\Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le \Lambda_\theta, \end{aligned} \qquad (44)$$

$$\begin{aligned} \Pi_\theta(\psi_\theta, c, k_\theta) \quad &= -\Pi(\psi_\theta, c, k_\theta) + \Pi(\psi_\theta(0), c, k_\theta) \\ &\qquad \text{for} \qquad 0 \le \lambda^{(\theta)} \le \Lambda_\theta/4 + \Lambda_\theta^{(2)}, \\ &= \Pi(\psi_\theta, c, k_\theta) + 2\Pi(c, k_\theta) + \Pi(\psi_\theta(0), c, k_\theta) \\ &\qquad \text{for} \qquad \Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le 3\Lambda_\theta/4 + \Lambda_\theta^{(2)}, \\ &= -\Pi(\psi_\theta, c, k_\theta) + 4\Pi(c, k_\theta) + \Pi(\psi_\theta(0), c, k_\theta) \\ &\qquad \text{for} \qquad 3\Lambda_\theta/4 + \Lambda_\theta^{(2)} \le \lambda^{(\theta)} \le \Lambda_\theta. \end{aligned} \qquad (45)$$

4.3. Consistency check of the analytical solution

In this subsection, we compare the analytical results of bound geodesics, $t(\lambda)$, $r(\lambda)$, $\cos\theta(\lambda)$ and $\phi(\lambda)$, with earlier literature as a consistency check. We can compare $r(\lambda)$ in (26) with that in [13, 22], in which the integral of motion, $\int \mathrm{d}r/\sqrt{R(r)} = \int \mathrm{d}\cos\theta/\sqrt{\Theta(\cos\theta)}$, is solved in terms of Jacobi's elliptic function using θ as the independent variable. Since [13] deals with null geodesics and [22] does not give explicit expressions including the turning points, we can not compare them exactly. However, we find that the formal expressions of $r(\lambda)$ in this paper and [13, 22] are consistent. We can also compare the formal expression of $\cos\theta(\lambda)$ in (38) with that in [13], in which the integral of motion is solved in terms of Jacobi's elliptic function using r as the independent variable. Then we find that the formal expressions of $\cos\theta(\lambda)$ in this paper and [13] are consistent. Although [13] derived t and ϕ in terms of Carlson elliptic integrals [15], it seems difficult to compare the expressions of both t and ϕ in this paper with that of null geodesics in [13]. Thus we compare the expressions of each integrals such as $\int_{r_2}^r (r')^2 \mathrm{d}r'/\sqrt{R(r')}$ in Appendix A with that in [15], in which formulas for $\int_y^x \prod_{i=1}^5 (a_i + b_i t)^{p_i/2} \mathrm{d}t$ are derived in terms of Carlson elliptic integrals, where all quantities are real, $x > y$ and $a_i + b_i t > 0$ for $y < t < x$. Using transformation of Carlson elliptic integrals of the third kind [23], it is straightforward to check $\int_{r_2}^r (r')^2 \mathrm{d}r'/\sqrt{R(r')}$ in Appendix A agrees with the corresponding formula in [15]. We can also check the

other integrals in this paper agree with that in [15]. These analytical checks show that the expressions in this paper are consistent.

Moreover, we compare the analytical results in this paper with that of numerical integration method. It is a good check on the results that both the coefficients and turning points in each integrals are consistent. In numerical integration of geodesic equation, as explained in section 2, we transform r as $r = pM/(1 + e \cos \psi)$ and $\cos \theta$ as $\cos \theta = \cos \theta_{\min} \cos \chi$ respectively. Using (2), we obtain the following set of differential equations [9]

$$\frac{d\psi}{d\lambda} = \frac{M\sqrt{(1 - \mathcal{E}^2)\{(p - p_3) - e(p + p_3 \cos \psi)\}\{(p - p_4) + e(p - p_4 \cos \psi)\}}}{1 - e^2},$$

$$\frac{d\chi}{d\lambda} = \sqrt{a^2(1 - \mathcal{E}^2)(z_+ - z_- \cos^2 \chi)},$$

$$\frac{dt}{d\lambda} = T_r(r) + T_\theta(\cos \theta) + a\mathcal{L}_z,$$

$$\frac{d\phi}{d\lambda} = \Phi_r(r) + \Phi_\theta(\cos \theta) - a\mathcal{E}, \tag{46}$$

where $p_3 = r_3(1 - e)/M$ and $p_4 = r_4(1 + e)/M$. We can numerically solve (46) accurately without taking account of the turning points of both r and $\cos \theta$ because both ψ and χ are monotonic increasing functions of time.

In figure 1, we compare the results of our analytical expressions with the results from the numerical integration method. We choose orbital parameters as $a = 0.9M$, $p = 4M$, $e = 0.7$ and $\theta_{\text{inc}} = 40°$. And set the initial values of ψ and χ as $\psi(0) = 0$ and $\chi(0) = 0$ respectively, which correspond to $r(0) = r_0^{(1)} = pM/(1 + e)$ and $\theta(0) = \theta_0^{(1)} = \pi/2$. For numerical integration of (46), we use the 4th order Runge-Kutta method with non-adaptive step-size control [17]. This figure shows that the analytical solutions of geodesic equation in this paper exactly represent the solutions of bound geodesic orbits around a Kerr black hole.

5. Summary

We have derived analytical solutions for bound timelike geodesics in Kerr spacetime. This is the first time that analytical expressions of the fundamental frequencies are derived in terms of the elliptic integrals. The analytical expressions of the orbits, $(t, r, \cos \theta, \phi)$, have been also derived in terms of the elliptic integrals using Mino time as the independent variable for the first time. Since Mino time decouples the r and θ-motion, it leads to forms simpler than that in [13] for null case if we suitably transform variables, r and θ. We checked the consistency of the analytical expressions comparing them with the analytical expressions for the other cases, post-Newtonian approximation and numerical integration method.

We can apply these solutions to the computation of gravitational waves from EMRIs. Gravitational waves from EMRIs are described by the Teukolsky formalism [24]. In the frequency domain calculation of the Teukolsky formalism [25, 26], we can use the analytical solutions directory [27, 28] and compute the orbits more accurately than
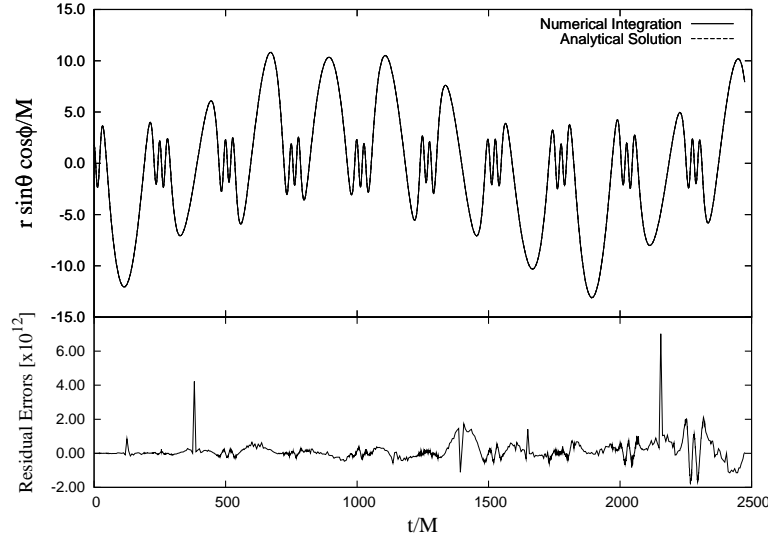
Figure 1. Comparison of the function $x(t) = r(t) \sin\theta(t) \cos\phi(t)$ computed using the analytical expressions of this paper with the result of a numerical integration. In this figure, we set orbital elements as $a = 0.9M$, $p = 4M$, $e = 0.7$ and $\theta_{\text{inc}} = 40°$. And we set initial value of ψ and χ as $\psi(0) = 0$ and $\chi(0) = 0$ respectively, which correspond to $r(0) = r_0^{(1)} = pM/(1 + e)$ and $\theta(0) = \theta_0^{(1)} = \pi/2$. Upper figure shows plots of both analytical solution, $x_{\text{A}}(t)$, and numerical integration method, $x_{\text{N}}(t)$. Lower figure shows the residual errors between the results of both analytical solution and numerical integration method, $x_{\text{A}}(t) - x_{\text{N}}(t)$.

numerical integration of geodesic equation. In principle, we can compute the orbits with machine accuracy. Using the analytical expressions of radial and polar motion in this paper, we showed that the analytical expressions enable one to compute gravitational waves from EMRIs very accurately [28]. Although it may takes longer time to compute the orbits using the analytical solutions than using numerical integration method [12], but see [13], it is not serious in computing gravitational waves. This is because we compute orbits only for one orbital period of radial and polar motion, Λ_r and Λ_θ, and computation time of the orbits, $\sim$seconds, is much smaller than that of gravitational waves from EMRIs, $\sim$hours to days [27, 28, 29]. Thus we believe that the analytical solutions are very useful for the computation of gravitational waves from EMRIs. In the time domain calculation of the Teukolsky formalism (see brief review in section 3.8 in [10]), we may need the inversion of $t(\lambda)$ in order to compute the orbits, $r(\lambda)$, $\cos\theta(\lambda)$, $t(\lambda)$ and $\phi(\lambda)$, in the coordinate time. Although we do not know the analytical expression of $\lambda(t)$, we may easily obtain $\lambda(t)$ by numerical iteration if we set the initial solution as $\lambda = t/\Gamma$. Thus it may also be useful in the time domain calculation if the numerical iteration converges faster than the numerical integration of the geodesic equation.

We may also apply these solutions to investigate the properties of geodesics of Kerr black holes. Although it seems difficult to classify orbits in the strong field because of its complexities, Levin *et al* recently suggested a taxonomy of orbits introducing

a rational number which is constructed from orbital frequencies [30, 31]. Both the analytical expressions of the fundamental frequencies in this paper and the taxonomy of orbits may help us to discuss the conditions characterizing zoom-whirl orbits [6] and other extreme phenomena in Kerr backgrounds. The other applications may be null or unbound geodesics. We can apply our method to them with a few modifications. For null geodesics, we have to eliminate the mass term of the small body in the geodesics. For unbound geodesics, we can express the orbits in terms of the elliptic integrals using Mino time although there are no fundamental frequencies for the orbits. However, it should be noted that we may have to improve computation time when we consider null or unbound geodesics because we have to trace the orbits for longer time than bound orbits cases. If we can not improve computation time, we may have to use both analytical solutions and numerical integration[12]. Finally, we note that we can not use the analytical solutions in this paper when the inclination angle from the equatorial plane of black hole is $\theta_{\mathrm{inc}} = \pi/2$. This is because $\Phi_\theta(\cos\theta)$ in (1) diverges when $\theta = 0$, and its elliptic integral also diverges. We do not know how to address this issue without any further approximation though one can solve it if one uses a post-Newtonian expansion [21]. All of them will be discussed in a future work.

Acknowledgments

Appendix A. formulas of integrals of radial motion

In this appendix, we derive some formulas which are needed to obtain $\Upsilon_{t^{(r)}}$ and $\Upsilon_{\theta^{(r)}}$ in section 3 and $t^{(r)}$ and $\phi^{(r)}$ in section 4. In order to compute them, we have to investigate $\int_{r_2}^r dr'/\{(r'-r_\pm)\sqrt{R(r')}\}$, $\int_{r_2}^r (r')^2 dr'/\sqrt{R(r')}$ and $\int_{r_2}^r dr'/\{(r'-M)^2\sqrt{R(r')}\}$, see (19). Since $\int_{r_2}^r r' dr'/\sqrt{R(r')}$ is derived in section 3 and $\int_{r_2}^r dr'/\{(r'-M)\sqrt{R(r')}\}$ can be derived from $\int_{r_2}^r dr'/\{(r'-r_\pm)\sqrt{R(r')}\}$ when we set $r_\pm = M$, i.e. $a = M$, we do not show again these expressions in this appendix.

   As we derived $\int_{r_2}^r r' dr'/\sqrt{R(r')}$ in section 3, it is useful to transform r into y_r. Then

we have the following relations.

$$\frac{1}{r - r_\pm} = \frac{1}{r_3 - r_\pm}\left[1 - \frac{r_2 - r_3}{r_2 - r_\pm}\frac{1}{1 - h_\pm y_r^2}\right],$$

$$r^2 = r_3^2 + \frac{(r_2 - r_3)(r_2 + 3r_3)}{2}\frac{1}{1 - h_r y_r^2} + \frac{(r_2 - r_3)^2}{4h_r}\left[\frac{1}{(y_r - h_r^{-1/2})^2} + \frac{1}{(y_r + h_r^{-1/2})^2}\right],$$
(A.1)

$$\left(\frac{r_3 - M}{r - M}\right)^2 = 1 + \frac{1}{2}\frac{r_2 - r_3}{r_2 - M}\left(\frac{r_2 - r_3}{r_2 - M} - 4\right)\frac{1}{1 - h_M y_r^2}$$

$$+ \frac{1}{4h_M}\left(\frac{r_2 - r_3}{r_2 - M}\right)^2\left[\frac{1}{(y_r - h_M^{-1/2})^2} + \frac{1}{(y_r + h_M^{-1/2})^2}\right],$$

where $h_M = h_\pm(a = M) = (r_1 - r_2)(r_3 - M)/[(r_1 - r_3)(r_2 - M)]$.

Then it is straightforward to compute $\int_{r_2}^{r} \mathrm{d}r'/\{(r' - r_\pm)\sqrt{R(r')}\}$ as

$$\int_{r_2}^{r} \frac{\mathrm{d}r'}{(r' - r_\pm)\sqrt{R(r')}} = \frac{2}{(r_3 - r_\pm)\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\left[F(\arcsin y_r, k_r)\right.$$

$$\left. - \frac{r_2 - r_3}{r_2 - r_\pm}\Pi(\arcsin y_r, -h_\pm, k_r)\right].$$
(A.2)

However, we need a reformulation of the last terms of both $\int_{r_2}^{r}(r')^2\mathrm{d}r'/\sqrt{R(r')}$ and $\int_{r_2}^{r}\mathrm{d}r'/\{(r' - M)^2\sqrt{R(r')}\}$. If we set $J_n[c] = \int_0^y \mathrm{d}y'/\{(y' - c)^n\sqrt{\varphi(y')}\}$, where $\varphi(y) = (1 - y^2)(1 - k_r^2 y^2)$, we can represent these terms as $J_2[c] + J_2[-c]$. Using reduction formula of the elliptic integrals, see section 17.1.5 in [14], we can express $J_2[c] + J_2[-c]$ in terms of the elliptic integrals as

$$J_2[c] + J_2[-c] = \frac{2}{\varphi(c)}\left\{\left[(2c^2 - 1)k_r^2 - 1\right]\Pi(\psi, -c^{-2}, k_r) + (1 - k_r^2 c^2)F(\psi, k_r) - E(\psi, k_r)\right.$$

$$\left. - \left[\frac{y\sqrt{\varphi(y)}}{y^2 - c^2}\right]_0^y\right\},$$
(A.3)

where $\psi = \arcsin y$.

Then we can express $\int_{r_2}^{r}(r')^2\mathrm{d}r'/\sqrt{R(r')}$ and $\int_{r_2}^{r}\mathrm{d}r'/\{(r' - M)^2\sqrt{R(r')}\}$ respectively as

$$\int_{r_2}^{r} \frac{(r')^2}{\sqrt{R(r')}}\mathrm{d}r' = \frac{2}{\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\left[\frac{(r_3(r_1 + r_2 + r_3) - r_1 r_2)}{2}F(\arcsin y_r, k_r)\right.$$

$$+ \frac{(r_2 - r_3)(r_1 + r_2 + r_3 + r_4)}{2}\Pi(\arcsin y_r, -h_r, k_r)$$

$$+ \frac{(r_1 - r_3)(r_2 - r_4)}{2}E(\arcsin y_r, k_r)$$

$$\left. + \frac{(r_1 - r_3)(r_2 - r_4)}{2}\frac{y_r\sqrt{(1 - y_r^2)(1 - k_r^2 y_r^2)}}{y_r^2 - h_r^{-1}}\right],$$
(A.4)

$$\int_{r_2}^{r} \frac{(r_3 - M)^2}{(r' - M)^2\sqrt{R(r')}}\mathrm{d}r' = \frac{2}{\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}$$

$$\times \left\{\left[1 - \frac{1}{2}\frac{(r_1 - r_3)(r_2 - r_3)}{(r_1 - M)(r_2 - M)}\right]F(\arcsin y_r, k_r)\right.$$

$$+ \frac{1}{2}\frac{r_2 - r_3}{r_2 - M}\left[\frac{r_1 - r_3}{r_1 - M} + \frac{r_2 - r_3}{r_2 - M} + \frac{r_4 - r_3}{r_4 - M} - 4\right]$$
$$\times \Pi(\arcsin y_r, -h_M, k_r)$$
$$+ \frac{1}{2}\frac{(r_1 - r_3)(r_2 - r_4)(r_3 - M)}{(r_1 - M)(r_2 - M)(r_4 - M)}E(\arcsin y_r, k_r)$$
$$+ \frac{1}{2}\frac{(r_1 - r_3)(r_2 - r_4)(r_3 - M)}{(r_1 - M)(r_2 - M)(r_4 - M)}\frac{y_r\sqrt{(1 - y_r^2)(1 - k_r^2 y_r^2)}}{y_r^2 - h_M^{-1}}\Bigg\} \quad \text{(A.5)}$$

Appendix B. $|a| = M$ case

In this appendix, we show the analytical expressions of Γ, Υ_ϕ $t^{(r)}$ and $\phi^{(r)}$ in the case $|a| = M$. Using the results of Appendix A, we derive them as

$$\Gamma = 4M^2\mathcal{E} + \frac{2a^2\mathcal{E}z_+\Upsilon_\theta}{\pi\mathcal{L}_z\sqrt{\epsilon_0 z_+}}[K(k_\theta) - E(k_\theta)]$$
$$+ \frac{2\Upsilon_r}{\pi\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}\Bigg\{\frac{\mathcal{E}}{2}[(r_3(r_1 + r_2 + r_3) - r_1 r_2)K(k_r)$$
$$+ (r_2 - r_3)(r_1 + r_2 + r_3 + r_4)\Pi(-h_r, k_r)$$
$$+(r_1 - r_3)(r_2 - r_4)E(k_r)] + 2M\mathcal{E}[r_3 K(k_r) + (r_2 - r_3)\Pi(-h_r, k_r)]$$
$$+ \frac{2M(4M^2\mathcal{E} - a\mathcal{L}_z)}{r_3 - M}\left[K(k_r) - \frac{r_2 - r_3}{r_2 - M}\Pi(-h_M, k_r)\right]$$
$$+ \frac{M^2(2M^2\mathcal{E} - a\mathcal{L}_z)}{(r_3 - M)^2}\left[\left(2 - \frac{(r_1 - r_3)(r_2 - r_3)}{(r_1 - M)(r_2 - M)}\right)K(k_r)\right.$$
$$+ \frac{(r_1 - r_3)(r_2 - r_4)(r_3 - M)}{(r_1 - M)(r_2 - M)(r_4 - M)}E(k_r)$$
$$+ \frac{r_2 - r_3}{r_2 - M}\left(\frac{r_1 - r_3}{r_1 - M} + \frac{r_2 - r_3}{r_2 - M} + \frac{r_4 - r_3}{r_4 - M} - 4\right)\Pi(-h_M, k_r)\Bigg]\Bigg\}, \quad \text{(B.1)}$$

$$\Upsilon_\phi = \frac{2\Upsilon_\theta}{\pi\sqrt{\epsilon_0 z_+}}\Pi(-z_-, k_\theta) + \frac{2a\Upsilon_r}{\pi\sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}}$$
$$\times \Bigg\{\frac{2M\mathcal{E}}{r_3 - M}\left[K(k_r) - \frac{r_2 - r_3}{r_2 - M}\Pi(-h_M, k_r)\right]$$
$$+ \frac{2M^2\mathcal{E} - a\mathcal{L}_z}{2(r_3 - M)^2}\left[\left(2 - \frac{(r_1 - r_3)(r_2 - r_3)}{(r_1 - M)(r_2 - M)}\right)K(k_r)\right.$$
$$+ \frac{(r_1 - r_3)(r_2 - r_4)(r_3 - M)}{(r_1 - M)(r_2 - M)(r_4 - M)}E(k_r)$$
$$+ \frac{r_2 - r_3}{r_2 - M}\left(\frac{r_1 - r_3}{r_1 - M} + \frac{r_2 - r_3}{r_2 - M} + \frac{r_4 - r_3}{r_4 - M} - 4\right)$$
$$\times\Pi(-h_M, k_r)]\Bigg\}, \quad \text{(B.2)}$$

$$t^{(r)} = \frac{2}{\sqrt{(1-\mathcal{E}^2)(r_1-r_3)(r_2-r_4)}}$$
$$\times \left\{ \frac{\mathcal{E}}{2} \left[(r_2-r_3)(r_1+r_2+r_3+r_4)\tilde{\Pi}_r(\psi_r,-h_r,k_r) \right.\right.$$
$$\left. + (r_1-r_3)(r_2-r_4)\tilde{E}_r(\psi_r,h_r,k_r) \right]$$
$$+ 2M\mathcal{E}(r_2-r_3)\tilde{\Pi}_r(\psi_r,-h_r,k_r) \qquad\qquad \text{(B.3)}$$
$$- \frac{2M(4M^2\mathcal{E}-a\mathcal{L}_z)}{r_3-M} \frac{r_2-r_3}{r_2-M}\tilde{\Pi}_r(\psi_r,-h_M,k_r)$$
$$+ \frac{M^2(2M^2\mathcal{E}-a\mathcal{L}_z)}{(r_3-M)^2} \left[\frac{(r_1-r_3)(r_2-r_4)(r_3-M)}{(r_1-M)(r_2-M)(r_4-M)}\tilde{E}_r(\psi_r,h_M,k_r) \right.$$
$$\left.\left. + \frac{r_2-r_3}{r_2-M}\left(\frac{r_1-r_3}{r_1-M}+\frac{r_2-r_3}{r_2-M}+\frac{r_4-r_3}{r_4-M}-4 \right)\tilde{\Pi}_r(\psi_r,-h_M,k_r) \right] \right\},$$

$$\phi^{(r)} = \frac{2a}{\sqrt{(1-\mathcal{E}^2)(r_1-r_3)(r_2-r_4)}} \left\{ -\frac{2M\mathcal{E}}{r_3-M}\frac{r_2-r_3}{r_2-M}\tilde{\Pi}_r(\psi_r,-h_M,k_r) \right.$$
$$+ \frac{2M^2\mathcal{E}-a\mathcal{L}_z}{2(r_3-M)^2} \left[\frac{(r_1-r_3)(r_2-r_4)(r_3-M)}{(r_1-M)(r_2-M)(r_4-M)}\tilde{E}_r(\psi_r,h_M,k_r) \right. \qquad \text{(B.4)}$$
$$\left.\left. + \frac{r_2-r_3}{r_2-M}\left(\frac{r_1-r_3}{r_1-M}+\frac{r_2-r_3}{r_2-M}+\frac{r_4-r_3}{r_4-M}-4 \right)\tilde{\Pi}_r(\psi_r,-h_M,k_r) \right] \right\}.$$

References

[1] Narayan R 2005 *New J. Phys.* **7** 199
[2] Chandrasekhar S 1983 *Mathematical theory of black holes* (Oxford: Oxford University Press)
[3] Wilkins D C 1972 *Phys. Rev.* D **5** 814
[4] Hughes S A 2001 *Phys. Rev.* D **63** 064016
[5] Barausse E, Hughes S A and Rezzolla L 2007 *Phys. Rev.* D **76** 044007
[6] Glampedakis K and Kennefick D 2002 *Phys. Rev.* D **66** 044002
[7] Schmidt W 2002 *Class. Quantum Grav.* **19** 2743
[8] Mino Y 2003 *Phys. Rev.* D **67** 084027
[9] Drasco S and Hughes S A 2004 *Phys. Rev.* D **69** 044015
[10] Glampedakis K 2005 *Class. Quantum Grav.* **22** S605
[11] *LISA web page* http://lisa.jpl.nasa.gov/
[12] Rauch K P and Blandford R D 1994 *Astrophys. J.* **421** 46
[13] Dexter J and Agol E 2009 *Astrophys. J.* **696** 1616
[14] Abramowitz M and Stegun I A (eds) 1972 *Handbook of Mathematical Functions* (New York: Dover)
[15] Carlson B C 1988 *Math. Comp.* **51** 267
[16] Bardeen J M 1973 *Black Holes* ed C DeWitt and B S DeWitt (New York: Gordon and Breach Science Publishers)
[17] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes in C* (Cambridge: Cambridge University Press)
[18] Bardeen J M, Press W H and Teukolsky S A 1972 *Astrophys. J.* **178** 347
[19] Cutler C, Kennefick D and Poisson E 1994 *Phys. Rev.* D **50** 3816
[20] Hughes S A 2000 *Phys. Rev.* D **61** 084004
      Hughes S A 2001 *Phys. Rev.* D **63** 049902(E)
      Hughes S A 2002 *Phys. Rev.* D **65** 069902(E)
      Hughes S A 2003 *Phys. Rev.* D **67** 089901(E)

[21] Ganz K, Hikida W, Nakano H, Sago N and Tanaka T 2007 *Prog. Theor. Phys.* **117** 1041
[22] Kraniotis G V 2007 *Class. Quantum Grav.* **24** 1775
[23] Zill D G and Carlson B C 1970 *Math. Comp.* **24** 199
[24] Teukolsky S A 1973 *Astrophys. J.* **185** 635
[25] Mino Y, Sasaki M, Shibata M, Tagoshi H and Tanaka T 1997 *Prog. Theor. Phys. Suppl.* **118** 1
[26] Sasaki M and Tagoshi H 2003 *Living Rev. Relativity* **6** 6
[27] Drasco S and Hughes S A 2006 *Phys. Rev.* D **73** 024027
[28] Fujita R, Hikida W and Tagoshi H 2009 *Prog. Theor. Phys.* **121** 843
[29] Hughes S A Drasco S Flanagan E E and Franklin J 2005 *Phys. Rev. Lett.* **94** 221101
[30] Levin J and Perez-Giz G 2008 *Phys. Rev.* D **77** 103005
[31] Levin J and Grossman B 2009 *Phys. Rev.* D **79** 043016